

Hyperplane mass equipartition problem and the shielding functions of Ramos*

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Abstract

We give a proof (based on methods and ideas developed in [16, 9, 18]) of the result of Ramos [11] which claims that two finite, continuous Borel measures μ_1 and μ_2 defined on \mathbb{R}^5 admit an equipartition by a collection of three hyperplanes. Our proof illuminates one of the central methods developed and used in our earlier papers and may serve as a good ‘test case’ for addressing (and resolving) the ‘issues’ raised in [2]; see Sections 1 and 4 for an outline and summary. We also offer a degree-theoretic interpretation of the ‘parity calculation method’ developed in [11] and demonstrate that, up to minor corrections or modifications, it remains a rigorous and powerful tool for proving results about mass equipartitions.

1 Introduction

The Grünbaum-Hadwiger-Ramos hyperplane mass equipartition problem [6, 7, 1, 11, 16, 9, 18, 19, 2, 3] has been for decades one of the important test problems for applications of topological methods in discrete geometry.

The problem came into the focus again with the appearance of the ‘critical review’ [2] which, as claimed by the authors, included the ‘documentation of essential gaps’ in the proofs of some of the earlier papers. In turn this led to an interesting and important academic discussion about the validity, scope and applicability of the previously used methods.

In this paper we address the central objections raised in [2] (the reader will find a brief summary in Section 1.1 and concluding comments in Section 4).

We begin with a proof of the result of Edgar Ramos [11] addressing the problem of equipartition of two finite, continuous Borel measures μ_1 and μ_2 defined on \mathbb{R}^5 by

*This paper is an updated and expanded version of [14].

a collection of three hyperplanes. The central idea of the proof (the ‘moment curve based evaluation of the topological obstruction’) originally appeared in one of our papers almost twenty years ago, see [16, Proposition 4.9].

As a corollary we obtain a new proof of the result $\Delta(1, 4) \leq 5$ (also due to Ramos) which states that each continuous measure in \mathbb{R}^5 admits an equipartition by 4 hyperplanes.

Despite the criticism and doubts raised in [2]¹, we prefer to interpret our evaluation $\Delta(2, 3) = 5$ as a different proof rather than the first complete proof of this result. As already emphasized, our proof illuminates one of the central methods developed in our earlier papers and we see it as a good ‘laboratory test case’ for discussing some of the ‘issues’ raised in [2].

In the second half of the paper (beginning with Section 3) we give an exposition of the Ramos ‘parity calculation method’ [11] emphasizing some of the key ideas, including the concept of the *shielding function* (Sections 2.3 and 3.2).

Detailed comments and concluding remarks, summarizing our current knowledge and opinion about the mass equipartition questions discussed in this paper, are collected in Section 4. Finally the Appendix (Section 5) is a short outline of fundamental ideas and facts about *transversality of equivariant maps* which should make the paper self-contained and easier to read.

Our ambition and the main objective in this paper was to address all main ‘issues’ raised by the authors of [2]. As it turned out there is actually only one central ‘issue’, related to the equivariant obstruction theory for non-free group actions. We shall demonstrate that the ‘issue’ disappears once we properly interpret the role of *shielding functions* introduced already by E. Ramos precisely for this purpose, the fact well known to the authors and many other experts in the field.

1.1 The CS/TM-scheme and the criticism of [2]

For the reader’s orientation here we place the criticism of [2] in the context of the general CS/TM-scheme for the mass equipartition problem. We hope that this outline may help the reader understand the main objection(s) of [2] and serve as an introduction to the rather obvious (and well known among specialists) remedy for the problem. In particular we emphasize the role of the ‘shielding functions’ which were originally introduced by Ramos in [11] precisely to avoid these difficulties.

A configuration space (I), the associated test space and the test map (II), and a topological result of Borsuk-Ulam type (III), are basic ingredients of the *Configuration space/test map method* (CS/TM-scheme) for applying equivariant topological methods in discrete geometry and combinatorics, see [16, 17] for an overview.

¹According to the ‘critical review’ [2, Table 2], the inequality $5 \leq \Delta(2, 3) \leq 8$ was the only available information about the number $\Delta(2, 3)$ at the time when the preprint [2] was submitted.

The general set-up for the CS/TM-scheme in the case of the mass equipartition problem was proposed by Ramos in [11]. He in particular identified proper configuration spaces and the test maps (steps (I) and (II)), which have been without essential change used in all subsequent publications.

In turn this led to the general agreement that the central difficulty in the problem is to establish the non-triviality of the associated topological obstruction (part (III)).

There have been two general methods to approach (III).²

- (A) The ‘parity count method’ of Ramos [11];
- (B) The ‘moment curve based evaluation of the topological obstruction’, introduced in [16] and subsequently developed in [9, 18].

The implementation of both of these methods was criticized in [2] and the authors of this paper claimed to have found ‘essential gaps’ in the proofs with a conclusion that *‘the approaches employed cannot work’* (see [2], the end of the page 2).

As the authors of some of the criticized papers, following the dictum that one should ‘consistently question one’s own findings’, we took these claims very seriously. Moreover, our professional curiosity was aroused and we wanted to understand the deeper nature of these claims.

Here is the summary of our response (more details can be found in Section 4 and elsewhere in the paper). We found that the criticism of [2] really applies to the ‘test map phase’ (step (II)) of the CS/TM-procedure and that it can be summarized as follows. The action of the symmetry group on the configuration space is not free. If the closed subspace of all singular orbits is removed, one obtains a truncated space (open manifold) where the topological obstruction is almost certainly equal to zero (and therefore an equivariant map should exist).

For illustration there certainly exists a \mathbb{Z}_2 -equivariant map $f : S^3 \setminus \{a, -a\} \rightarrow \mathbb{R}^3$ without zeros, i.e. the Borsuk-Ulam theorem is no longer true if one removes two antipodal points from the domain. However, if the map f can be extended to a \mathbb{Z}_2 -equivariant map $g : S^3 \rightarrow \mathbb{R}^3$ without zeros in $\{a, -a\}$ (or if it is equivariantly homotopic to such a map) then the Borsuk-Ulam theorem holds for f as well.

In practise f is already defined on the whole configuration space and it is guaranteed that it has no zeros in the singular set by the *shielding functions* (called the ‘shield functions’ by Ramos in [11]).

E. Ramos was fully aware of this technical difficulty and the shielding functions were introduced by him in [11] precisely for this purpose. The reader is referred to Section 2.3 and Section 3.2 for a more detailed explanation of the importance of shielding functions and their role in methods (A) and (B).

In summary, with this clarification, both the methods (A) and (B) are fully applicable and the proofs and results obtained by their application are correct.

²The methods applied in [19] are based on somewhat different ideas so its presentation is postponed for a subsequent publication.

2 Mass equipartitions by hyperplanes

The reader is referred to [11, 16, 9, 19, 2] for an overview of known results and the history of the general measure equipartition problem by hyperplanes. Recall that the problem has its origins in the papers of Grünbaum [6] and Hadwiger [7], with the papers of Steinhaus [12] and Stone and Tukey [13] as important predecessors.

2.1 CS/TM-scheme for the mass equipartition problem

A collection $\mathcal{A} = \{A_1, \dots, A_j\}$ of Lebesgue measurable sets in \mathbb{R}^d admits an *equipartition* by a collection $\mathcal{H} = \{h_1, \dots, h_k\}$ of hyperplanes if $m(A_i \cap O) = (1/2^k)m(A_i)$ for each $i = 1, \dots, j$ and each of the 2^k hyperorthants O associated to \mathcal{H} .

More generally a collection $\mathcal{M} = \{\mu_1, \dots, \mu_j\}$ of continuous, finite, Borel measures defined on \mathbb{R}^d admits an equipartition by \mathcal{H} if $\mu_i(O) = 1/2^k \mu_i(\mathbb{R}^d)$ for each $i = 1, \dots, j$.

The ‘equipartition number’ $\Delta(j, k)$ is defined as the minimum dimension d of the ambient space \mathbb{R}^d such that each collection \mathcal{M} of j continuous measures admits an equipartition by some collection \mathcal{H} of k hyperplanes. We also say that a triple (d, j, k) is *admissible* if $\Delta(j, k) \leq d$.

Following [11] and [9] the compactified *configuration space* for the general mass equipartition problem is the manifold $M_{d,k} = (S^d)^k$ where $h = (h_1, \dots, h_k) \in M_{d,k}$ is an ordered k -tuple of oriented hyperplanes (including the hyperplanes ‘at infinity’).

Given a 0-1-sequence (alternatively a $(+ -)$ -sequence) $J = (j_1 j_2 \dots j_k) \in 2^{[k]}$ and a k -tuple $h \in M_{d,k}$, the associated half-spaces are $h_1^{j_1}, \dots, h_k^{j_k}$ and the *test function* $a_J^\mu(h) := \mu(\bigcap_{\nu=1}^k h_\nu^{j_\nu})$ measures the amount of mass μ in the corresponding hyperorthant. Let $\hat{J} = \{\nu \in [k] \mid j_\nu = 1\}$ be the subset of $[k]$ determined by J .

Following [11] the collection $\{a_J^\mu \mid J \in 2^{[k]}\}$ of test functions is (via a Discrete Fourier Transform) replaced by the functions,

$$f_I^\mu(h) = f_{i_1 \dots i_k}^\mu(h_1, \dots, h_k) = \sum_{J \in 2^{[k]}} (-1)^{\langle I, J \rangle} a_J^\mu(h) \quad (1)$$

where $\langle I, J \rangle = i_1 j_1 + \dots + i_k j_k = |\hat{I} \cap \hat{J}|$ is the cardinality of the set $\hat{I} \cap \hat{J}$.

Remark 1. The reader should observe that in the CS/TM-scheme described here we tacitly use the continuity properties of measures when we extend functions a_J^μ and f_I^μ to the whole configuration space $M_{d,k}$ (which includes hyperplanes ‘at infinity’).

2.2 $\Delta(2, 3) = 5$

The equipartition problem attracted new audience and received wider recognition with the appearance of the paper of E. Ramos [11] who introduced new technique and obtained many new results about the function $d = \Delta(j, k)$ including the following,

$$4 \leq \Delta(1, 4) \leq 5 \quad \Delta(3, 2) = 5 \quad 7 \leq \Delta(3, 3) \leq 9 \quad \Delta(4, 2) = 6. \quad (2)$$

Among the most interesting is his claim that $\Delta(2, 3) = 5$ which allowed him to prove, by a simple reduction, that $\Delta(1, 4) \leq 5$.

Here we give a different proof of a slightly more general result. The generalization may be of some independent interest, however it primarily exemplifies the phenomenon that a strengthened statement may be sometimes easier to prove, cf. [16, Proposition 4.9] for an early example in the context of equipartitions of masses by hyperplanes.

Recall (Section 2.1) that if h is an oriented hyperplane then for $\epsilon \in \{+, -\}$ the associated closed half-space is denoted by h^ϵ .

Theorem 2. *Suppose that μ_1, μ_2, μ_3 are three continuous, finite, non-negative Borel measures defined on \mathbb{R}^5 . Then there exist three hyperplanes h_1, h_2, h_3 in \mathbb{R}^5 forming an equipartition for measures μ_1 and μ_2 such that one of them (h_i for some $i \in [3]$) is a bisector of μ_3 in the sense that $\mu_3(h_i^+) = \mu_3(h_i^-)$.*

Proof: The theorem says that in addition to being an equipartition for the measures μ_1, μ_2 it can be always achieved that one of the hyperplanes h_1, h_2, h_3 is a halving hyperplane for a third measure μ_3 (which is also prescribed in advance).

Following the usual *configuration space/test map scheme*, see [9] and Section 2.1, the configuration space (parameterizing all triples (h_1, h_2, h_3) of oriented hyperplanes in \mathbb{R}^5 including the hyperplanes ‘at infinity’) is $M = S^5 \times S^5 \times S^5$. The group of symmetries acting on M by permuting the hyperplanes and changing their orientation is the group $G = \mathbb{Z}_2^{\oplus 3} \rtimes S_3$. This group arises also as the group of symmetries of the 3-dimensional cube.

The real regular representation $\mathbb{R}[\mathbb{Z}_2^{\oplus 3}]$ of $\mathbb{Z}_2^{\oplus 3}$ is also a G -representation. The test space V_i for each of the measures μ_i , $i = 1, 2$ is the G -representation of dimension 7 arising by subtracting from $\mathbb{R}[\mathbb{Z}_2^{\oplus 3}]$ the trivial 1-dimensional representation,

$$V_i = \left\{ \sum_{\epsilon} \alpha_{\epsilon} \cdot \epsilon \in \mathbb{R}[\mathbb{Z}_2^{\oplus 3}] : \sum_{\epsilon} \alpha_{\epsilon} = 0 \right\}.$$

We also need a copy of \mathbb{R} to serve as the target space for testing if one of the hyperplanes is a bisector for μ_3 , so the total test space is the G -representation $V = V_1 \oplus V_2 \oplus \mathbb{R}$.

In agreement with (1) the associated ‘test map’,

$$f = (f_1, f_2, f_3) : S^5 \times S^5 \times S^5 \rightarrow V_1 \oplus V_2 \oplus \mathbb{R} \quad (3)$$

is described as follows. Since the hyperplanes h_i are oriented each hyperorthant $O_{\epsilon} = O_{(\epsilon_1, \epsilon_2, \epsilon_3)} = h_1^{\epsilon_1} \cap h_2^{\epsilon_2} \cap h_3^{\epsilon_3}$ is associated an element $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$ of the group $\mathbb{Z}_2^{\oplus 3}$. By definition (for $i = 1, 2$),

$$f_i(h_1, h_2, h_3) = \sum_{\epsilon} [\mu_i(O_{\epsilon}) - \frac{1}{8} \mu_i(\mathbb{R}^5)] \cdot \epsilon \in V_i \subset \mathbb{R}[\mathbb{Z}_2^{\oplus 3}] \quad (4)$$

The map $f_3 : S^5 \times S^5 \times S^5 \rightarrow \mathbb{R}$ is defined by,

$$f_3(h_1, h_2, h_3) = (\mu_3(h_1^+) - \mu_3(h_1^-))(\mu_3(h_2^+) - \mu_3(h_2^-))(\mu_3(h_3^+) - \mu_3(h_3^-)) \quad (5)$$

By construction a triple $h = (h_1, h_2, h_3)$ satisfies the conditions of the theorem if and only if h is a zero of the test map (3). The test map is clearly G -equivariant so it is sufficient to show that there does not exist a G -equivariant map $f : M \rightarrow V \setminus \{0\}$ (arising from measures, as in the construction above).

Remark 3. If $\mu_1(\mathbb{R}^5) = \mu_2(\mathbb{R}^5) = 0$ then there is a ‘trivial’ solution of the equation $f(h) = 0$. Indeed, if H is a bisector of both μ_1 and μ_2 then $h = (H, H, H)$ is a clearly an equipartition for $\{\mu_1, \mu_2\}$. For this reason we tacitly assume (throughout the paper) that either $\mu_1(\mathbb{R}^d) > 0$ or $\mu_2(\mathbb{R}^d) > 0$ (or both).

2.3 Shielding functions

The action of G on $M = S^5 \times S^5 \times S^5$ is not free. The singular set

$$S = \bigcup_{x, y \in S^5} (G \cdot (x, x, y) \cup G \cdot (x, -x, y))$$

consists of ordered triples having two equal or two antipodal elements. In light of Remark 3 we are allowed to assume that $\mu_i(\mathbb{R}^5) = \lambda > 0$ for at least one $i \in \{1, 2\}$, say for $i = 1$. For most of the interesting measures (including positive measures) both of these two conditions are satisfied.

As an immediate consequence we deduce that $f(S) \subset V \setminus \{0\}$. Moreover, all maps $f = f_{(\mu_1, \mu_2, \mu_3)} : S \rightarrow V \setminus \{0\}$ arising from measures such that $\mu_1(\mathbb{R}^5) > 0$ **are linearly G -homotopic**. Indeed, the value of $f_1(x, x, y) \in \mathbb{R}[\mathbb{Z}_2^{\oplus 3}]$ has a non-zero coefficient $-(1/8)\lambda$ associated to the element $\epsilon = (+, -, \epsilon_3)$ while the value of $f_i(x, -x, y)$ has the same non-zero coefficient at $\epsilon = (+, +, \epsilon_3)$. This is (in light of the definition of the test map (4)) a simple consequence of the fact that whenever a hyperorthant is empty the corresponding coefficient is equal to $-(1/8)\mu_i(\mathbb{R}^5) \neq 0$.

Let $g : S \rightarrow V \setminus \{0\}$ be a representative of this G -homotopy class, for example g can be chosen to be the restriction of a map f_{ν_1, ν_2, ν_3} for a particular choice of measures ν_1, ν_2, ν_3 . Theorem 2 will be proved if we show that the first (and only) obstruction class $\omega = \omega_g$ for extending equivariantly the map g to $M = S^5 \times S^5 \times S^5$ is non-zero.

Definition 4. (Shielding functions) *The functions that keep the zeros of the test map away from the singular set (where the action of the group is not free) are following Ramos [11] called the shielding or shield functions. The fact that all test maps arising from measures are non-zero and G -homotopic on the singular set is referred to as the ‘shielding functions homotopy principle’.*

2.4 Calculation of the obstruction ω

The obstruction ω described in the previous section lives in the relative equivariant cohomology group $H_G^{15}(M, S; W)$ where W is the G -module $\pi_{14}(V \setminus \{0\})$. By taking a small G -equivariant open tubular neighborhood U of S we observe that ω can be evaluated in the group $H_G^{15}(N, \partial N; W)$ where $N = M \setminus U$ is a compact G -manifold with boundary.

In light of the equivariant Poincaré-Lefschetz duality this group is isomorphic to the equivariant homology group $H_0^G(N, W \otimes \pi)$ where π is the orientation character describing the action of G on the (relative) fundamental class of N . This group is isomorphic to one of the groups \mathbb{Z}_2 or \mathbb{Z} and the corresponding dual of ω can be evaluated in this group by a geometric argument.

Note that since the action of G on the manifold with boundary N is free, by passing to the quotient manifold N/G we can actually use the usual version of Poincaré-Lefschetz duality (with local coefficients). However there is a shortcut which allows us to complement (bypass) the homological algebra related to the (equivariant) Poincaré-Lefschetz duality and replace it by a direct geometric argument.³

Proposition 5. *Suppose that $f : M \rightarrow V$ is a G -equivariant extension of $g : S \rightarrow V \setminus \{0\}$. Moreover, assume that f is smooth outside of a small tubular neighborhood U of S ($g(U) \subset V \setminus \{0\}$) and that f is transverse to $0 \in V$. The set $f^{-1}(0)$ is finite and G -invariant. The number of G -orbits $n_f = |f^{-1}(0)/G|$ depends on f , however the parity of this number $\theta = \theta_f =_{\text{mod } 2} n_f \in \mathbb{Z}_2$ is the same for all extensions f of g . In particular if this number is odd then each G -extension of g must have a zero.*

Proof: The result is an immediate consequence of Proposition 22. □

In order to compute the obstruction $\theta \in \mathbb{Z}_2$ we use the map $f : M \rightarrow V$ which arises from the following choice of measures (measurable sets). Let Γ be the moment curve in \mathbb{R}^5 defined as the image $\phi(\mathbb{R})$ of the map $\phi : \mathbb{R} \rightarrow \mathbb{R}^5, t \mapsto (t, t^2, \dots, t^5)$. Suppose that I_1, I_2, I_3 are three disjoint, consecutive intervals on this curve (Figure 1) and let ν_i be the measures on \mathbb{R}^5 defined by $\nu_i(A) = m(\phi^{-1}(A \cap I_i))$.

By construction $h = (h_1, h_2, h_3) \in f^{-1}(0)$ is the set of triples of oriented hyperplanes in \mathbb{R}^5 such that h is an equipartition for both ν_1 and ν_2 and in addition one of the hyperplanes h_i is a bisector of ν_3 . Altogether there are at most 15 points in the set $\Gamma \cap (h_1 \cup h_2 \cup h_3)$ and all fifteen are needed (Figure 1) if $h \in f^{-1}(0)$.

For bookkeeping purposes let us analyze possible ‘types’ of elements $h = (h_1, h_2, h_3) \in f^{-1}(0)$. We begin with the analysis which cardinalities of sets $h_i \cap I_j$ are permitted. A closer inspection reveals that there are only three possibilities (Figure 2) and we notice that only some pairs of ‘complementary types’ (associated to intervals I_1 and I_2) can appear together.

³Here we follow the same general strategy applied in [9], both in the overall technical set-up and in the computational details (which are of course much more complex in the case $(8, 5, 2)$ studied in the paper [9]).



Figure 1: Three hyperplanes intersect the moment curve in 15 points.

For example one of the possibilities, abbreviated as $h = \langle (4_1, 2_2, 1_3), (1_1, 3_2, 3_3) \rangle$ or simply as $h = \langle (4, 2, 1), (1, 3, 3) \rangle$, describes the case where the cardinalities of the intersections are,

$$|h_1 \cap I_1| = 4 \quad |h_2 \cap I_1| = 2 \quad |h_3 \cap I_1| = 1$$

and

$$|h_1 \cap I_2| = 1 \quad |h_2 \cap I_2| = 3 \quad |h_3 \cap I_2| = 3.$$

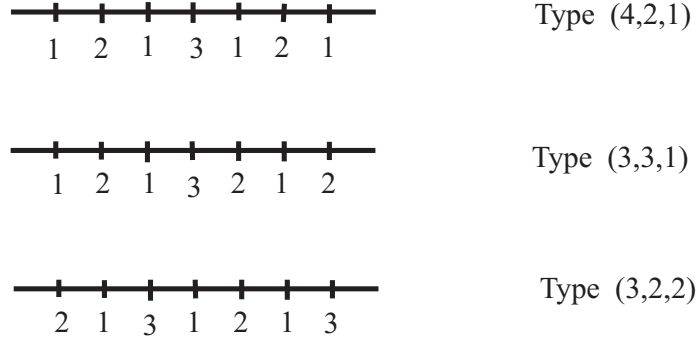


Figure 2: Up to a permutation of $\{1, 2, 3\}$ there are three possible types of equipartitions of an interval on the moment curve by three hyperplanes.

All permutations of indices are possible, for example in the same G -orbit with h is the element $h' = \langle (4_3, 2_1, 1_2), (1_3, 3_1, 3_2) \rangle = \langle (2, 1, 4), (3, 3, 1) \rangle$.

The following Claim summarizes the information needed for the evaluation of the number of G -orbits in $f^{-1}(0)$. Our initial observation is that in each orbit $G \cdot h \subset f^{-1}(0)$ there is an element $h' = (h_1, h_2, h_3)$ such that the type of the intersection $(h_1 \cup h_2 \cup h_3) \cap I_1$ is precisely one of the types listed in Figure 2.

Claim:

- (1) The I_1 -type $(4, 2, 1)$ can be matched with the I_2 -type $(1, 2, 4)$ in only one way, contributing **1** orbit;
- (2) The I_1 -type $(4, 2, 1)$ can be matched with the I_2 -type $(1, 3, 3)$ in two ways, contributing **2** orbits;
- (3) The I_1 -type $(3, 3, 1)$ can be uniquely matched with both $(1, 2, 4)$ and $(2, 1, 4)$ (as the I_2 -types), which together contribute **2** orbits;

- (4) The I_1 -type $(3, 3, 1)$ can be matched with the I_2 -type $(2, 2, 3)$ in two ways, contributing **2** orbits;
- (5) The I_1 -type $(3, 2, 2)$ can be matched with the I_2 -type $(1, 3, 3)$ in two ways, contributing **2** orbits;
- (6) The I_1 -type $(3, 2, 2)$ can be matched with the I_2 -type $(2, 3, 2)$ in two ways, contributing **2** orbits;
- (7) The I_1 -type $(3, 2, 2)$ can be matched with the I_2 -type $(2, 2, 3)$ in two ways, contributing **2** orbits.

For illustration let us check the case (3). We can uniquely choose $h = (h_1, h_2, h_3)$ in this orbit so that the I_1 -type is precisely the middle type shown in Figure 2. Using this information we reconstruct the intersection of hyperplanes h_i with the interval I_3 . If the type of this intersection is $(1, 2, 4)$ the midpoint of I_2 belongs to h_1 while the intersection $h_2 \cap I_2$ has two elements which also uniquely determines their positions in I_2 . The case when the I_2 -type is $(2, 1, 4)$ is treated similarly. The other cases listed in the Claim are checked by a similar reasoning.

Altogether there are 13 G -orbits in the set $f^{-1}(0)$ which shows that the parity of the obstruction is $\theta = 1$.

For the completion of the proof of Theorem 2 we should convince ourselves that all zeros $h \in f^{-1}(0)$ are non-degenerate. This is established along the lines of the proof of Theorem 33 in [9], see also the comments on the proof of Theorem 4 on page 291 (ibid.). The proof in our case is actually much simpler since we are interested in the parity calculation, i.e. we don't have to worry about the sign of the Jacobian matrix.

For a chosen $h = (h_1, h_2, h_3) \in f^{-1}(0)$ and an arbitrary triple $l = (L_1, L_2, L_3)$ in a small neighborhood U of h we observe that l is determined by the fifteen numbers,

$$\{x_1 < x_2 < \dots < x_{15}\} = (L_1 \cup L_2 \cup L_3) \cap (I_1 \cup I_2 \cup I_3)$$

which therefore can be used as coordinating functions on U .

If $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ and $I_3 = [a_3, b_3]$ then the hyperplanes L_1, L_2, L_3 divide the interval I_1 in hyperorthants $[a_1, x_1], [x_1, x_2], \dots, [x_7, b_1]$ and the interval I_2 in hyperorthants $[a_2, x_8], [x_8, x_9], \dots, [x_{14}, b_2]$. From here we observe that the functions,

$$x_1 - a_1, x_2 - x_1, \dots, x_7 - x_6 \quad \text{and} \quad x_8 - a_2, x_9 - x_8, \dots, x_{14} - x_{13}$$

can be used as coordinates on the (truncated) test space $V_1 \oplus V_2$. By an affine change of coordinates we see that x_1, x_2, \dots, x_{14} can be used as the coordinates on the space $V_1 \oplus V_2$ as well. From here and the fact that $\partial f_3 / \partial x_{15} \neq 0$ we easily conclude that the corresponding Jacobian matrix is non-singular. \square

Corollary 6. *Each continuous measure μ in \mathbb{R}^5 admits an equipartition by 4 hyperplanes. Moreover one of these hyperplanes can be chosen to be a common bisector of 4 measurable sets (measures) prescribed in advance, and one of the remaining hyperplanes is also a bisector of a chosen measurable set.*

Proof: The case $(5, 1, 4)$ (one measure and the equipartition by 4 hyperplanes) is reduced to the case $(5, 2, 3)$ (two measures and the equipartition by 3 hyperplanes) by taking a bisector H of μ and applying Theorem 2 to the two new measures μ_+, μ_- defined by $\mu_+(A) = \mu(H^+ \cap A)$ and $\mu_-(A) = \mu(H^- \cap A)$. By the ‘ham sandwich theorem’ (applied in \mathbb{R}^5) the hyperplane H can be chosen as a halving hyperplane of four additional measurable sets. Similarly by Theorem 2 one of the remaining hyperplanes can be chosen as a bisector for a chosen measurable set. \square

2.5 Other cases of the equipartition problem

The method applied in our proof of Theorem 2 is not new, indeed it has been developed in our papers and over the years successfully applied to many cases of the equipartition problem.

Ramos in [11] isolated the triples $(4, 1, 4)$, $(8, 5, 2)$ and $(7, 3, 3)$ as the first cases not covered by his method. For this reason we focused and tested our method initially in these cases.

The admissibility of the triple $(8, 5, 2)$ was established in [9]. At the same time we calculated (following the same plan as in the proof of Theorem 2) the obstruction in the case of the triple $(7, 3, 3)$. The result turned out to be zero and this is the reason why this observation was not published, although it was reported in our lectures and presentations of the (much more complicated) case $(8, 5, 2)$.

Essentially the same strategy was applied by the authors of [3] who established (as the only new result of the paper) the admissibility of the triple $(10, 4, 3)$. Moreover it is not difficult to observe that, as long as the calculations are dependent on the ‘moment curve based evaluation of the topological obstruction’ (method (B)), as described in Sections 1.1 and 2.4), the ‘join scheme’ and the ‘product scheme’ described in [3, Section 1.2.] are equivalent! In other words the ‘join scheme’ cannot bring anything new that is not already provable by the ‘product scheme’.

Finally, the paper [19] used different methods to establish the admissibility of the triple $(6 \cdot 2^\nu + 2, 4 \cdot 2^\nu + 1, 2)$ for each $\nu \geq 0$, which includes $(8, 5, 2)$ as a special case. This paper also came under criticism of [2] and these objections, being of somewhat different nature, will be addressed in our subsequent publication.

3 The method of Ramos as developed in [11]

The paper of Edgar Ramos [11] introduced important new ideas into the mass equipartition problem, leading to the algorithms for calculating relevant topological obstructions. Indeed, this paper has been for years an inspiration for all subsequent work in this area.

The authors of [2] acknowledge the importance of [11], however they claim that the proofs of central results of this paper have essential gaps. If proved correct, the critical

analysis from [2] would render obsolete not only the proofs of the results from [11] but the ideas and the methods would be also affected. This in particular applies to the ‘parity count’ formulas from [11] which would be probably avoided in the future, if the criticism from [2] is taken for granted.

This would be a shame since these methods (and proofs) are (essentially) correct in a strong sense of the word. They continue to be a valuable tool for tackling problems in this and related areas of *Applied and computational algebraic topology*.

In the following section we revisit, reprove and give a slightly different interpretation to the parity count lemmas and formulas from [11]. In particular we explain why the ‘counterexample’ [2, Section 7] is not properly addressing Lemma 6.2 from [11] which remains correct and applicable.

3.1 The parity count formulas from [11]

E. Ramos used in [11] intricate parity count calculation (in the setting of piecewise linear topology) to evaluate the obstruction to the existence of equipartitions of masses by hyperplanes. His approach was critically analyzed in [2] and some of his results were doubted, in particular the authors of [2] questioned the validity of his proof of the equality $\Delta(2, 3) = 5$.

We begin this exposition by observing that the parity count formulas of Ramos [11, p. 150], designed for evaluating the ‘parity invariants’ $P(r; X)$ and $P^+(r', \underline{r}''; X)$, can be naturally interpreted as formulas/algorithms for evaluating the degrees of associated maps.

Definition 7. Suppose that $r : (X, \partial(X)) \rightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ is a map where X is a compact, n -dimensional manifold with boundary $\partial(X)$. Then by definition,

$$P(r; X) := \deg\{H_n(X, \partial(X); \mathbb{Z}_2) \xrightarrow{r^*} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}_2)\} \quad (6)$$

is the mod_2 -degree of the map r .

If X is a smooth or triangulated manifold such that 0 is not a critical value of r (i.e. r is transverse to zero $r \pitchfork \{0\}$), then $P(r; X)$ is indeed the parity (the mod_2 -cardinality) of the (finite) set $r^{-1}(0)$.

Definition 8. Suppose that $s : Y \rightarrow \mathbb{R}^n \setminus \{0\}$ is a map defined on a compact, $(n-1)$ -dimensional manifold without boundary. Then by definition,

$$P(s; Y) := \deg\{H_{n-1}(Y; \mathbb{Z}_2) \xrightarrow{s^*} H_{n-1}(\mathbb{R}^n \setminus \{0\}; \mathbb{Z}_2)\}. \quad (7)$$

Proposition 9. Choose $v \in \mathbb{R}^n \setminus \{0\}$ and let $L = \{\lambda v \mid \lambda \geq 0\}$ be the associated half-ray in \mathbb{R}^n . Suppose that $s : Y \rightarrow \mathbb{R}^n \setminus \{0\}$ is a map defined on a smooth (alternatively PL-triangulated) compact, $(n-1)$ -dimensional manifold without boundary. Assume that s is transverse to the ray L , $s \pitchfork L$, and let $P(s, L; Y)$ be the mod_2 -cardinality of the set $s^{-1}(L)$. Then,

$$P(s; Y) = P(s, L; Y). \quad (8)$$

Proof: Let $\psi : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$ be the radial projection, $v \mapsto v/\|v\|$. Then $s \pitchfork L$ if and only if $v/\|v\|$ is a regular value of the map $\psi \circ s$. The result follows from the well-known fact the mod_2 -degree of a map $g : Y \rightarrow S^{n-1}$ can be calculated as the mod_2 -cardinality of the set $g^{-1}(a)$ for any regular value $a \in S^{n-1}$. \square

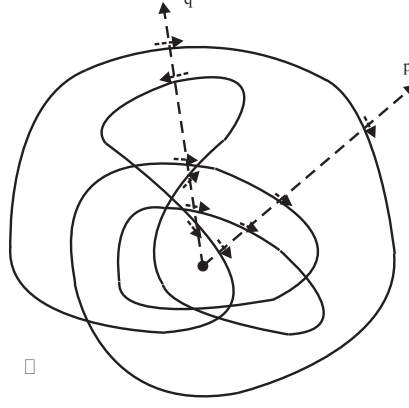


Figure 3: The winding number as the number of signed intersections with a half-ray.

Remark 10. The formula (8) in the planar case reduces to the (mod_2 -version) of the well-known description of the winding number of a curve as the number of signed intersections with a (generic) half-ray (Figure 3).

The following ‘standard genericity assumption’ summarizes the conditions needed for comparison of different parity calculations (as in [11, Lemma 2.1.]).

Definition 11. (Standard genericity assumption) *We say that a map $r = (r', r'') : (X, \partial(X)) \rightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$, defined on a compact, n -dimensional manifold X with boundary $\partial(X)$, satisfies the **standard genericity assumption** if:*

- (1) *X is a smooth (alternatively PL-triangulated) manifold such that 0 is not a critical value of r .*
- (2) *The restriction $s = r|_{\partial(X)}$ of r on the boundary $\partial(X)$ of X is transverse to L where $L = \{0\} \times \mathbb{R}^+ \subset \mathbb{R}^{n-1} \times \mathbb{R}$.*

Here r' denotes the first $(n-1)$ -components of r and r'' is the last component of r while L is by definition the positive semi-axis corresponding to the last coordinate in \mathbb{R}^n .

Proposition 12. ([11, Lemma 2.1.]) *Let $r = (r', r'') : (X, \partial(X)) \rightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ be a map defined on a compact, n -dimensional manifold X with boundary $\partial(X)$ which satisfies the ‘standard genericity assumption’ in the sense of Definition 11. Then,*

$$P(r, X) = P^+(r', \underline{r}''; \partial(X)) \quad (9)$$

where by definition $P^+(r', \underline{r}''; \partial(X)) = P(s, L; \partial(X))$.

Proof: The equality of mapping degrees $P(r; X)$ (formula (6)) and $P(s; \partial(X))$ (formula (7)) is well-known, so the equality (9) is an immediate consequence of Proposition 9. \square

3.2 Shielding functions revisited

Shielding functions are (under the name ‘shield functions’) explicitly mentioned in [11] at least three times. However the references to these functions are ubiquitous in the paper. As already emphasized in Section 2.3 the role of these functions is to *shield* the zeros of the equipartition test function from appearing in the singular set S where the action of the group is not free. More explicitly, in agreement with Definition 4, f is a shielding function for $A \subset S$ if $f(x) \neq 0$ for each $x \in A$. The following proposition, cf. [11, Property 4.5.(i)], describes an important class of shielding functions. Recall that for a given 0-1-sequence $I \in 2^{[k]}$ the associated set is $\hat{I} = \{\nu \in [k] \mid i_\nu = 1\}$.

Proposition 13. ([11, Property 4.5.(i)]) *Let $I = (i_1 i_2 \dots i_k) \in 2^{[k]}$ and suppose that the cardinality of the set $\hat{I} = \{\nu \in [k] \mid i_\nu = 1\}$ is even. Suppose that h_1, \dots, h_k are oriented hyperplanes such that $h_{i_{\nu_1}} = h_{i_{\nu_2}}$ if $\{\nu_1, \nu_2\} \subset \hat{I}$. Then f_I^μ is a shielding function in the sense that,*

$$f_I^\mu(h) = f_{i_1 \dots i_k}^\mu(h_1, \dots, h_k) = 1 \quad (10)$$

Proof: From the assumption that $h_{i_{\nu_1}} = h_{i_{\nu_2}}$ for each pair $\{\nu_1, \nu_2\} \subset \hat{I}$ we deduce that $a_J(h)$ can be non-zero only in two cases, if either $\hat{J} \cap \hat{I} = \emptyset$ or $\hat{J} \cap \hat{I} = \hat{I}$. By definition $f_I(h) = \sum_J (-1)^{|\hat{I} \cap \hat{J}|} a_J(h)$ so if $|\hat{I} \cap \hat{J}|$ is odd (knowing that $|\hat{I}|$ is even) then $\emptyset \neq \hat{I} \cap \hat{J} \neq \hat{I}$ and as a consequence $a_J(h) = 0$. \square

Remark 14. The most important is the case when \hat{I} has two elements. For example if $\hat{I} = \{1, 2\}$ then the associated shielding function is $f_{110\dots 0}^\mu$. Observe that we do not need new shielding functions to shield other singular points in the configuration space where the action is not free. Indeed, if $h_i = -h_j$ for some pair of indices $\{i, j\} \subset [k]$ than the same function (10) that shields the region where $h_i = h_j$ can be used again. For example if $i = 1$ and $j = 2$ then $f_{110\dots 0}^\mu(h) = -1$ if $h_1 = -h_2$.

3.3 Recursive computation of the parity number

The following result from [11] is central tool for the recursive computation of the parity number $P(r, X)$.

Proposition 15. ([11, Theorem 2.2.]) *Let $r = (r', r'') : X \rightarrow \mathbb{R}^{n-1} \times \mathbb{R}$ be a function satisfying all the conditions listed in Proposition 12, including the ‘standard genericity assumption’.*

- (i) *Suppose that $\partial(X) = \bigcup_{i=1}^s Y_i$ where Y_i are pairwise, interior disjoint subcomplexes, $\text{int}(Y_i) \cap \text{int}(Y_j) = \emptyset$ for $i \neq j$. Assume that $\text{int}(Y_i)$ are open manifolds and that r' has no zeros in the union of all boundaries $\bigcup_{i=1}^s (Y_i \setminus \text{int}(Y_i))$. Then,*

$$P(r, X) = \sum_{i=1}^s P^+(r', \underline{r}''; Y_i). \quad (11)$$

- (ii) Let $Z_{r'} = r'^{-1}(0)$ be the zero-set of the function r' . Suppose that $Y_i = Y_{i,1} \cup Y_{i,2}$ is an interior disjoint union such that r' has no zeros in the set $Y_{i,\epsilon} \setminus \text{int}(Y_{i,\epsilon})$ (for $\epsilon \in \{0, 1\}$). Assume that there exists a bijection $\beta : Y_{i,1} \cap Z_{r'} \rightarrow Y_{i,2} \cap Z_{r'}$ (usually a restriction of a homeomorphism $\beta : Y_{i,1} \rightarrow Y_{i,2}$) such that for some integer a , $r''(\beta(x)) = (-1)^a r''(x)$ for each $x \in Y_{i,1} \cap Z_{r'}$. Then,

$$P^+(r', \underline{r}''; Y_i) = a \cdot P(r'; Y_{i,1}). \quad (12)$$

Proof: Both statements are immediate consequences of Proposition 12. \square

Remark 16. In order to apply Proposition 15 one should be able to guarantee that there are no zeroes of the function r' in any of the sets $Y_i \setminus \text{int}(Y_i)$ (respectively $Y_{i,\epsilon} \setminus \text{int}(Y_{i,\epsilon})$). This is usually achieved by one of the following requirements.

- (1) The map r' is generic and $\dim(Y_i \setminus \text{int}(Y_i)) < \dim(Y_i)$;
- (2) A component of r' is a *shielding function*.

Example 17. As an illustration we demonstrate how the Borsuk-Ulam theorem follows from the parity calculation described in Proposition 15. To this end we show that if $f : B^n \rightarrow \mathbb{R}^n$ is a (generic) map, which is \mathbb{Z}_2 -equivariant in the sense that $f(-x) = -f(x)$ for each $x \in S^{n-1}$, then $P(f, B^n) = 1$. Since the action is free here we do not need shielding functions, i.e. the condition (1) from Remark 16 is sufficient. If $f = (f', f'') : B^n \rightarrow \mathbb{R}^{n-1} \times \mathbb{R}$ then by Proposition 15,

$$P(f, B^n) = P^+(f', \underline{f}''; S^{n-1}) = P^+(f', \underline{f}''; S_+^{n-1}) + P^+(f', \underline{f}''; S_-^{n-1})$$

where S_+^{n-1} and S_-^{n-1} are the hemispheres. Since $f''(-x) = -f''(x)$ for $x \in S^{n-1}$ it follows from Proposition 15(ii) that $P(f, B^n) = P(f', S_+^{n-1})$. Since $P(g, B^1) = 1$ for a function $g : [-1, 1] \rightarrow \mathbb{R}$ satisfying the condition $g(-1) = -g(1)$ the proof is completed by induction.

Example 18. In this example we review the proof of $\Delta(2, 2) = 3$ based on [11, Section 5.1.] (following the notation from this paper). We outline some of the key steps illuminating the role of the shielding functions.

The inequality $\Delta(2, 2) \leq 3$ is deduced from $P(\mathbf{r}, (B^2)_{\leq}^2) = 1$ where

$$(B^2)_{\leq}^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq \|x_1\| \leq \|x_2\| \leq 1\}$$

and

$$\mathbf{r} = (r_{11}^{\mu_1}, r_{01}^{\mu_2}, r_{10}^{\mu_2}, r_{11}^{\mu_2}) : (B^2)_{\leq}^2 \rightarrow \mathbb{R}^4 \quad (13)$$

is the associated test-map. The calculation begins with the observation that,

$$P(\mathbf{r}, (B^2)_{\leq}^2) = P^+(\underline{r}_{11}^{\mu_1}, r_{01}^{\mu_2}, r_{10}^{\mu_2}, r_{11}^{\mu_2}; X_{1,2}) + P^+(\underline{r}_{11}^{\mu_1}, r_{01}^{\mu_2}, r_{10}^{\mu_2}, r_{11}^{\mu_2}; X_{2,3}) \quad (14)$$

where $X_{1,2}$ (respectively $X_{2,3}$) are the subsets of $(B^2)_{\leq}^2$ satisfying the condition $\|x_1\| = \|x_2\|$ (respectively $\|x_2\| = 1$). Here we use the full power of Remark 16, especially the

fact that a component $r_{11}^{\mu_2}$ of the (reduced) test map $\mathbf{r}' = (r_{01}^{\mu_2}, r_{10}^{\mu_2}, r_{11}^{\mu_2})$ is a shielding function. The author continues by showing that,

$$P^+(\underline{r_{11}^{\mu_1}}, r_{01}^{\mu_2}, r_{10}^{\mu_2}, r_{11}^{\mu_2}; X_{2,3}) = 0 \quad (15)$$

which relies on the fact that for each $(x_1, x_2) \in X_{2,3}$,

$$\mathbf{r}'(x_1, x_2) = 0 \iff \mathbf{r}'(x_2, x_1) = 0 \quad \text{and} \quad r_{11}^{\mu_1}(x_1, x_2) = r_{11}^{\mu_1}(x_2, x_1).$$

Here he again uses the fact that $r_{11}^{\mu_2}$ is a shielding function which in light of Proposition 13 rules out the possibility $\mathbf{r}'(x, x) = 0$.

Remark 19. It is interesting to compare two possible ways of justifying the parity calculation in Example 18. In our approach we would prefer to keep the equivariance and sacrifice the transversality of the test map on the singular set. This is always possible as demonstrated in Section 5. Ramos would prefer to preserve transversality and offer as a sacrifice the equivariance of the test map on the singular set (or in its vicinity). In both approaches it is the presence of the shielding function $r_{11}^{\mu_2}$ that keeps the zeroes of \mathbf{r}' away from the troublesome region. Both approaches are correct and lead to correct calculations.

3.4 ‘Counterexample’ to [11, Lemma 6.2.]

Generalizing the calculations used in his proof of $\Delta(2, 2) = 3$ (outlined in our Example 18) Ramos formulated and proved the following proposition (Lemma 6.2. in [11, Section 6]). Our formulation is identical to the original with the addition of the condition (tacitly assumed throughout the whole of section Section 6 in [11]) that for each of the symmetries involved a component of the reduced test map \mathbf{r}' is a *shielding function* for the associated singular set. (The meaning of the phrase ‘*symmetric for zeros on the boundary*’ is explained in [11, page 162], prior to Lemma 6.2.)

Proposition 20. ([11, Lemma 6.2.]) *Suppose that $\mathbf{r} = (\mathbf{r}', \underline{r''}) : (B^n)_{\leq}^k \rightarrow \mathbb{R}^{nk}$ is a NPL (non-degenerate piecewise linear) map which is symmetric for zeros in the boundary and let $a \in \{0, 1\}$ be the antipodality character of r'' with respect to the k -th ball. Assume that for each symmetry β involved there is a component of \mathbf{r}' acting as a shielding function for the region $S_\beta = \{x \mid \beta(x) = x\}$. Then,*

$$P(\mathbf{r}', r''; (B^n)_{\leq}^k) = a \cdot P(\mathbf{r}'; (B^n)_{\leq}^{k-1} \times B^{n-1}). \quad (16)$$

Proof: Following [11] the lemma is a direct consequence of Proposition 15. \square

The key objection of [2] to the ‘parity calculation method’ of Ramos was summarized and exemplified by their ‘counterexample’ to his [11, Lemma 6.2] (our Proposition 20), see Example 7.7 in [2, p. 22].

We claim that this ‘counterexample’ is not correct in the sense that it does not address properly Lemma 6.2. Explicitly, the map $r = (r', r'') : (B^1)_{\leq}^3 \rightarrow \mathbb{R}^3$ they

describe does not satisfy the condition that sufficiently many components of the reduced map r' are shielding functions!⁴

Nevertheless let us take a look at their argument more closely. The authors of [2] summarize their objection by saying that their example ‘*exploits the simple fact that the permutation action on the coordinates in $C_{m;n}$ has fixed points, a fact that Ramos does not account for in his proof*’.

This assertion is clearly incorrect since the whole concept of a shielding function (shield function) was invented for this purpose. A quick computer search through [2] reveals that the word ‘*shield*’ (as in ‘shield function’) is completely absent from their paper, in particular it is not mentioned in their ‘counterexample’ to Lemma 6.2. It appears that the authors of [2] unfortunately did not read [11] carefully enough and apparently missed to observe the central role played by shielding functions in this paper.

An objective reader may correctly remark that after all there is a missing condition in [11, Lemma 6.2.]. We could agree with this to some extent, however this is hardly an ‘essential gap’ leading to the conclusion that the ‘approach employed cannot work’.

Moreover, at the end of the paragraph (typeset in the fine print) immediately after the proof of Lemma 6.2. the reader will find the following lines:

Thus, it is correct to assume that the NPL (non-degenerate piecewise linear) approximations have the required symmetry properties as long as in the expansion in which symmetry is used, a *shield function remains for each symmetry used*.

In other words Ramos reiterates the importance of shielding functions and formulates precisely the ‘missing condition’ from his Lemma 6.2.

4 Our response to [2]

The paper [2] is welcome as an invitation to an interesting and important problem in geometric combinatorics, however it leaves much to be desired on the level of careful and accurate presentation and interpretation of earlier proofs and results.

This is a pity since the criticism is always welcome, as it provides an opportunity to improve the presentation and test one’s overall understanding of the problem.

We agree that the exposition in all papers [11, 16, 9, 18] can be improved, notably in our papers [16, 9, 18] we tacitly used the assumption that all test maps arise from measures (see Sections 2.3 and 4.1).

However we strongly disagree with the negative conclusions from [2]. We claim that the insight, basic constructions and the results in [11, 16, 9, 18] are correct. The same applies to the paper [19] (a detailed analysis of the methods used in this paper is postponed for a subsequent paper).

⁴This is evident already from the fact that the zeroes of r' are in the union of boundary sets $F_{x,y}, F_{y,z}, F_{x,z}$ which all should be shielded since they consist solely of singular points.

4.1 The ‘gaps’ and ‘corrigenda’

The reader may wonder how is it possible that the ‘essential gaps’ in so many papers passed unnoticed until the appearance of [2]. The answer is there are no ‘essential gaps’ in these papers. Here we offer a footnote size ‘corrigendum’ summarizing what was said about ‘shielding functions’ in previous sections.

- (1) All equivariant test maps (see our Section 2.1 and [11, Section 4]) arise from measures. As a consequence if some of the hyperplanes coincide some of the hyperorthants are degenerated and have measure zero.
- (2) The test maps, restricted to the singular set (where some hyperplanes coincide) are therefore linearly homotopic (the ‘shielding functions homotopy principle’ (Definition 4 in Section 2.3)).
- (3) One uses the relative, rather than the absolute equivariant obstruction theory, as explicitly suggested already in our original paper [16, Remark 4.3] (see also the introductory part of Section 2.4).

The reader may ask what is the main content of papers [16, 9, 18] (if the assumptions (1)–(3) are tacitly treated there as part of the overall (technical) set-up). The answer is that the real challenge in the Grünbaum-Hadwiger-Ramos hyperplane mass partition problem is always the concrete **evaluation** of the topological obstruction. Here are some highlights.

1. The central new idea introduced in [16] and developed in [9, 18] is to use measures supported by the moment curve for the evaluation of the obstruction. Together with the ‘parity count method’ of Ramos this is still the only general method used by all papers including [3].
2. The use of the moment curve (or any other *convex curve*) reduces the evaluation of the obstruction to a problem of enumerative combinatorics, namely to enumeration of combinatorial patterns related to Hamiltonian paths in hypercubes (Gray codes), see our Figure 1 and compare it to Figure 2 in [16] or Figures 2 and 3 in [9].
3. The central fact leading to the main result in [18] (Theorem 5.1) was the observation that the *unique balanced 4-bit Gray code* has an inner symmetry (Figures 2, 3, and 4 in [18]).

4.2 Equivariant cobordism and shielding functions

There is another important idea (point of view), more or less explicit in [16, 9, 18] (see for example Section 2.3. in [18]), which also involves shielding functions and explains to some extent how it happened that (1)–(3) (from Section 4.1) were not more explicitly stated among the assumptions in these papers.

The idea explains how one can justify the use of open manifolds, for example (as in our Section 2.3) the use of the configuration space $M_\delta = (S^5)_\delta^3 := (S^5)^3 \setminus S$ obtained by removing the singular orbits. The illustrative case of 2-equipartitions of a single measure in \mathbb{R}^2 is presented in [18, Section 2.3.] as an introduction and motivation for the more interesting (and complex) ‘symmetric 4-dimensional case’ (treated later in the same paper). The following details are extracted from [18, Section 2.3.].

The open manifold $M_\delta = (S^2)_\delta^2$ parameterizes pairs of distinct, oriented (affine) lines in \mathbb{R}^2 . Note that M_δ is an open, free \mathbb{D}_8 -manifold where \mathbb{D}_8 is the dihedral group of order 8. Given a (sufficiently regular) continuous measure μ_0 on \mathbb{R}^2 , the associated solution set $\Sigma_{\mu_0} \subset (S^2)_\delta^2$ of all equipartitions of μ_0 is a compact 1-dimensional manifold (equipped with a free action of \mathbb{D}_8). A (sufficiently generic) path $\{\mu_t\}_{t \in [0,1]}$ (homotopy), connecting μ_0 to any other (generic) measure μ_1 , defines an equivariant cobordism N_μ between the solution sets Σ_{μ_0} and Σ_{μ_1} .

Remark 21. The manifold N_μ is a compact surface! This may appear obvious and (on second thought) is obvious, however note that precisely here we rely on the fact that the equipartitions of a given family $\{\mu_t\}_{t \in [0,1]}$ of measures cannot escape to infinity. Formally this is guaranteed by the existence of the corresponding shielding function!

The proof is completed by showing that the obstruction cobordism class $\theta = [\Sigma_{\mu_0}]$ represents the generator in the group $\Omega_1(\mathbb{D}_8) \cong \mathbb{Z}_4$ of equivariant bordisms. This is done by choosing the unit disc D^2 as the test measure μ_0 or alternatively (which is more suitable for generalizations) its boundary S^1 which is an example of a *convex curve* in \mathbb{R}^2 .

4.3 The ‘gaps’ in the paper [11] of Ramos

On the basis of the analysis presented in Section 3, see in particular Section 3.4, we conclude that there are no gaps in the paper [11] of Ramos. Moreover his ‘parity count method’ remains a rigorous and powerful tool for proving results about equipartitions of masses by hyperplanes.

4.4 The role of shielding functions in [2]

As observed in Section 3.4 a computer search through [2] shows that the word ‘*shield*’ or ‘*shielding*’ (function) is completely absent from [2]. It appears that the authors of [2] completely overlooked one of the key technical ingredients in all papers [11, 16, 9, 18]. This may explain why they spent a lot of time and energy in [2, Section 6] discussing (counter)examples which have little to do with actual methods used in [11, 16, 9, 18].

Note that these rather technical examples (see [2, Section 6]) are hardly surprising to experts interested in generalizations of the Borsuk-Ulam theorem. For example in [2, Theorem 6.1.] they establish the existence of a Σ_k^\pm -equivariant map $Z_{d,k} \rightarrow S(U_k^{\oplus j})$ and in particular an equivariant map $(S^4)_\delta^4 \rightarrow S(U_4)$. This is more complicated but otherwise similar in spirit to the ‘Borsuk-Ulam example’ introduced in Section 1.1 claiming that there exists a \mathbb{Z}_2 -equivariant map from $S_\delta^3 = S^3 \setminus \{a, -a\}$ to $\mathbb{R}^3 \setminus \{0\}$.

5 Appendix

In this section we collect some basic, equivariant transversality lemmas used throughout the paper. Our primary objective is to provide an overview and some technical details needed for the proof of Proposition 5 and for the applications in Section 3. The exposition is elementary and fully accessible to a non-expert, including a combinatorially minded reader without much previous exposure to algebraic topology.

5.1 Equivariant transversality theorem

Let G be a finite group and suppose that M is a n -dimensional, compact, smooth G -manifold. Let $S \subset M$ be the ‘singular set’ $S = \{x \in M \mid G_x \neq e\}$ of points with a non-trivial stabilizer.

Suppose that V is a real, n -dimensional representation of G and let $g : S \rightarrow V \setminus \{0\}$ be a continuous, G -equivariant map.

Suppose that $f : M \rightarrow V$ is a G -equivariant extension of g which are transverse to $\{0\} \in V$. The set $f^{-1}(0)$ is finite and G -invariant. The number of G -orbits $n_f = |f^{-1}(0)|/|G|$ clearly depends on f . The following elementary lemma claims that the parity of this number $\theta = \theta_f =_{\text{mod } 2} n_f \in \mathbb{Z}_2$ is the same for all extensions f of g in the same relative G -homotopy class.

Proposition 22. *Suppose that $f_0, f_1 : M \rightarrow V$ are two G -equivariant extensions of g which are transverse to $\{0\} \in V$. Then,*

$$\theta(f_0) \equiv \theta(f_1) \pmod{2}. \quad (17)$$

Proof: Let $F : M \times [0, 1] \rightarrow V$ be the linear homotopy between f_0 and f_1 defined by $F(x, t) = (1 - t)f_0(x) + tf_1(x)$. Let $A = (S \times [0, 1]) \cup (M \times \{0, 1\})$. The map F is by assumption already transverse to $\{0\} \in V$ on the set A . Since the action of the group G is free on $(M \times I) \setminus (S \times I)$, we are allowed to apply the ‘Equivariant Transversality Theorem’ (Theorem 23) which claims that the homotopy F admits a small G -equivariant perturbation,

$$H : M \times [0, 1] \rightarrow V \quad (18)$$

which is transverse to $0 \in V$ everywhere. Moreover we can assume that H and F agree on the set A . It follows that $H^{-1}(0)$ is a compact, 1-dimensional manifold (G -bordism), connecting zero sets $Z(f_0)$ and $Z(f_1)$. In turn there is a bordism between the associated sets $Z(f_0)/G, Z(f_1)/G$ of orbits and the equality (17) is an immediate consequence. \square

Theorem 23. (Equivariant Transversality Theorem) *Let G be a finite group and suppose that N is a n -dimensional, compact, smooth G -manifold. Let $S \subset N$ be the*

‘singular set’ $S = \{x \in N \mid G_x \neq e\}$ of points with a non-trivial stabilizer. Suppose that V is a real, k -dimensional representation of G . Suppose that,

$$F : N \rightarrow V$$

is a G -map such that $0 \notin F(S)$. Suppose that $A \subset N$ is a closed G -subset where F is already transverse to $\{0\}$. Then there exists a G -equivariant map,

$$H : N \rightarrow V$$

transverse to $\{0\}$ which agrees with F on $A \cup S$. Moreover, there exists a relative G -homotopy (small perturbation of F),

$$G : N \times [0, 1] \rightarrow V \quad (\text{rel } A \cup S)$$

connecting F and H .

Comments on the proof of Theorem 23: Theorem 23 is quite directly a consequence of the standard (non-equivariant) transversality theorem (or its proof), as exposed in [5, 8] and other textbooks.

One of the guiding principles used in the standard proofs (see for example [5]) is to make the map transverse locally (one small open set at a time) using the fact that a small perturbation will not affect the transversality condition achieved earlier on in the construction.

This can be done equivariantly, in the region where the action is *free*. Indeed, if $V \subset N$ is a (small) open set such that $V \cap g(V) = \emptyset$ for each $g \neq e$ then F can be made transverse to $\{0\}$ on V and extended equivariantly to $\cup_{g \in G} g(V)$, etc.

Another possibility is to use the fact that equivariant maps are sections of a bundle. Let U be the interior of a G -invariant regular neighborhood of S . Let $N' = N \setminus U$ and $\partial N' = \partial \overline{U}$. We can assume that $0 \notin F(\partial N')$. Then the action of G on N' is free and there is a one-to-one correspondence between G -equivariant maps $f : N' \rightarrow V$ and sections of the bundle,

$$V \longrightarrow N' \times_G V \rightarrow N'/G. \quad (19)$$

Moreover f is transverse to $\{0\}$ if and only if s is transverse (in the usual, non-equivariant sense) to the zero section of the bundle. \square

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